

HOLOMORPHIC HARMONIC MORPHISMS FROM FOUR-DIMENSIONAL NON-EINSTEIN MANIFOLDS

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ABSTRACT. We construct 4-dimensional Riemannian Lie groups carrying left-invariant conformal foliations with minimal leaves of codimension 2. We show that these foliations are holomorphic with respect to an (integrable) Hermitian structure which is *not* Kähler. We then prove that the Riemannian Lie groups constructed are *not* Einstein manifolds. This answers an important open question in the theory of complex-valued harmonic morphisms from Riemannian 4-manifolds.

1. INTRODUCTION

More than twenty years ago, J. C. Wood proved, in [7], that any submersive harmonic morphism from an orientable 4-dimensional Einstein manifold M to a Riemann surface, or a conformal foliation of M by minimal surfaces, determines an (integrable) Hermitian structure with respect to which it is holomorphic. Ever since it has been an open question whether holomorphicity in the above situation forces the 4-manifold M to be Einstein.

In this paper we construct two 3-dimensional families of 4-dimensional Riemannian Lie groups carrying left-invariant conformal foliations with minimal leaves of codimension 2. We show that these foliations are holomorphic with respect to an (integrable) Hermitian structure which is *not* Kähler. We then prove that the Riemannian Lie groups constructed are *not* Einstein manifolds. This gives a definite answer to the above mentioned open question.

For the general theory of harmonic morphisms between Riemannian manifolds we refer to the excellent book [2] and the regularly updated on-line bibliography [4].

2. HARMONIC MORPHISMS AND MINIMAL CONFORMAL FOLIATIONS

Let M and N be two manifolds of dimensions m and n , respectively. A Riemannian metric g on M gives rise to the notion of a *Laplacian* on (M, g) and real-valued *harmonic functions* $f : (M, g) \rightarrow \mathbb{R}$. This can be generalized to the concept of *harmonic maps* $\phi : (M, g) \rightarrow (N, h)$ between Riemannian

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manifolds, which are solutions to a semi-linear system of partial differential equations, see [2].

Definition 2.1. A map $\phi : (M, g) \rightarrow (N, h)$ between Riemannian manifolds is called a *harmonic morphism* if, for any harmonic function $f : U \rightarrow \mathbb{R}$ defined on an open subset U of N with $\phi^{-1}(U)$ non-empty, $f \circ \phi : \phi^{-1}(U) \rightarrow \mathbb{R}$ is a harmonic function.

The following characterization of harmonic morphisms between Riemannian manifolds is due to Fuglede and T. Ishihara. For the definition of horizontal (weak) conformality we refer to [2].

Theorem 2.2. [3, 6] *A map $\phi : (M, g) \rightarrow (N, h)$ between Riemannian manifolds is a harmonic morphism if and only if it is a horizontally (weakly) conformal harmonic map.*

Let (M, g) be a Riemannian manifold, \mathcal{V} be an involutive distribution on M and denote by \mathcal{H} its orthogonal complement distribution on M . As customary, we also use \mathcal{V} and \mathcal{H} to denote the orthogonal projections onto the corresponding subbundles of TM and denote by \mathcal{F} the foliation tangent to \mathcal{V} . The second fundamental form for \mathcal{V} is given by

$$B^{\mathcal{V}}(U, V) = \frac{1}{2}\mathcal{H}(\nabla_U V + \nabla_V U) \quad (U, V \in \mathcal{V}),$$

while the second fundamental form for \mathcal{H} satisfies

$$B^{\mathcal{H}}(X, Y) = \frac{1}{2}\mathcal{V}(\nabla_X Y + \nabla_Y X) \quad (X, Y \in \mathcal{H}).$$

The foliation \mathcal{F} tangent to \mathcal{V} is said to be *conformal* if there is a vector field $V \in \mathcal{V}$ such that

$$B^{\mathcal{H}} = g \otimes V,$$

and \mathcal{F} is said to be *Riemannian* if $V = 0$. Furthermore, \mathcal{F} is said to be *minimal* if $\text{trace } B^{\mathcal{V}} = 0$ and *totally geodesic* if $B^{\mathcal{V}} = 0$. This is equivalent to the leaves of \mathcal{F} being minimal and totally geodesic submanifolds of M , respectively.

It is easy to see that the fibres of a horizontally conformal map (resp. Riemannian submersion) give rise to a conformal foliation (resp. Riemannian foliation). Conversely, the leaves of any conformal foliation (resp. Riemannian foliation) are locally the fibres of a horizontally conformal map (resp. Riemannian submersion), see [2].

The next result of Baird and Eells gives the theory of harmonic morphisms, with values in a surface, a strong geometric flavour.

Theorem 2.3. [1] *Let $\phi : (M^m, g) \rightarrow (N^2, h)$ be a horizontally conformal submersion from a Riemannian manifold to a surface. Then ϕ is harmonic if and only if ϕ has minimal fibres.*

3. 4-DIMENSIONAL LIE GROUPS

Let G be a 4-dimensional Lie group equipped with a left-invariant Riemannian metric. Let \mathfrak{g} be the Lie algebra of G and $\{X, Y, Z, W\}$ be an orthonormal basis for \mathfrak{g} . Let $Z, W \in \mathfrak{g}$ generate a 2-dimensional left-invariant and integrable distribution \mathcal{V} on G which is conformal and with minimal leaves. We denote by \mathcal{H} the horizontal distribution, orthogonal to \mathcal{V} , generated by $X, Y \in \mathfrak{g}$. Then it is easily seen that the basis $\{X, Y, Z, W\}$ can be chosen so that the Lie bracket relations for \mathfrak{g} are of the form

$$\begin{aligned} [W, Z] &= \lambda W, \\ [Z, X] &= \alpha X + \beta Y + z_1 Z + w_1 W, \\ [Z, Y] &= -\beta X + \alpha Y + z_2 Z + w_2 W, \\ [W, X] &= aX + bY + z_3 Z - z_1 W, \\ [W, Y] &= -bX + aY + z_4 Z - z_2 W, \\ [Y, X] &= rX + \theta_1 Z + \theta_2 W \end{aligned}$$

with real structure constants. For later reference we state the following easy result describing the geometry of the situation.

Proposition 3.1. *Let G be a 4-dimensional Lie group and $\{X, Y, Z, W\}$ be an orthonormal basis for its Lie algebra as above. Then*

- (i) \mathcal{F} is totally geodesic if and only if $z_1 = z_2 = z_3 + w_1 = z_4 + w_2 = 0$,
- (ii) \mathcal{F} is Riemannian if and only if $\alpha = a = 0$, and
- (iii) \mathcal{H} is integrable if and only if $\theta_1 = \theta_2 = 0$.

On the Riemannian Lie group G there exist, up to sign, exactly two invariant almost Hermitian structure J_1 and J_2 which are adapted to the orthogonal decomposition $\mathfrak{g} = \mathcal{V} \oplus \mathcal{H}$ of the Lie algebra \mathfrak{g} . They are determined by

$$\begin{aligned} J_1 X &= Y, \quad J_1 Y = -X, \quad J_1 Z = W, \quad J_1 W = -Z, \\ J_2 X &= Y, \quad J_2 Y = -X, \quad J_2 W = Z, \quad J_2 Z = -W. \end{aligned}$$

An elementary calculation involving the Nijenhuis tensor shows that J_1 is integrable if and only if

$$(3.1) \quad 2z_1 - z_4 - w_2 = 2z_2 + z_3 + w_1 = 0.$$

In this case the Lie bracket relations for \mathfrak{g} take the form

$$\begin{aligned} [W, Z] &= \lambda W, \\ [Z, X] &= \alpha X + \beta Y + z_1 Z - (2z_2 + z_3)W, \\ [Z, Y] &= -\beta X + \alpha Y + z_2 Z + (2z_1 - z_4)W, \\ [W, X] &= aX + bY + z_3 Z - z_1 W, \\ [W, Y] &= -bX + aY + z_4 Z - z_2 W, \\ [Y, X] &= rX + \theta_1 Z + \theta_2 W. \end{aligned}$$

Example 3.2. With the non-vanishing coefficients z_2, θ_1, θ_2 we obtain the following 3-dimensional family of Lie algebras

$$\begin{aligned} [Z, X] &= -2z_2W, \\ [Z, Y] &= z_2Z, \\ [W, Y] &= -z_2W, \\ [Y, X] &= 2z_2X + \theta_1Z + \theta_2W. \end{aligned}$$

These are the special cases $\mathfrak{g}_7(z_2, -2z_2, 0, \theta_1, \theta_2)$ of Example 5.3 in [5]. It should be noted that the horizontal distribution \mathcal{H} is not integrable and the leaves of the vertical foliation \mathcal{V} are not totally geodesic.

We will show that none of the corresponding Riemannian Lie groups are Einstein manifolds. A standard calculation involving the Koszul formula

$$2\langle \nabla_X Y, Z \rangle = \langle [Z, X], Y \rangle + \langle [Z, Y], X \rangle + \langle Z, [X, Y] \rangle$$

shows that the Levi-Civita connection satisfies the following relations

$$\begin{aligned} \nabla_X X &= 2z_2Y, \quad \nabla_X Y = -2z_2X - \frac{1}{2}\theta_1Z - \frac{1}{2}\theta_2W, \\ \nabla_X Z &= \frac{1}{2}\theta_1Y + z_2W, \quad \nabla_X W = \frac{1}{2}\theta_2Y - z_2Z, \\ \nabla_Y X &= \frac{1}{2}\theta_1Z + \frac{1}{2}\theta_2W, \quad \nabla_Y Y = 0, \\ \nabla_Y Z &= -\frac{1}{2}\theta_1X, \quad \nabla_Y W = -\frac{1}{2}\theta_2X, \\ \nabla_Z X &= \frac{1}{2}\theta_1Y - z_2W, \quad \nabla_Z Y = -\frac{1}{2}\theta_1X + z_2Z, \\ \nabla_Z Z &= -z_2Y, \quad \nabla_Z W = z_2X, \\ \nabla_W X &= \frac{1}{2}\theta_2Y - z_2Z, \quad \nabla_W Y = -\frac{1}{2}\theta_2X - z_2W, \\ \nabla_W Z &= z_2X, \quad \nabla_W W = z_2Y. \end{aligned}$$

This means that the Hermitian structure J_1 is not Kähler, since

$$(\nabla_X J_1)(X) = -\frac{1}{2}(\theta_1Z + \theta_2W).$$

Employing the definition for the sectional curvature

$$\langle R(X, Y)Y, X \rangle = \langle \nabla_X \nabla_Y Y - \nabla_Y \nabla_X Y - \nabla_{[X, Y]} Y, X \rangle$$

we then obtain the following useful equalities

$$\begin{aligned} \langle R(X, Y)Y, X \rangle &= -\frac{3}{4}(\theta_1^2 + \theta_2^2) - 4z_2^2, \quad \langle R(X, Z)Z, X \rangle = \frac{1}{4}\theta_1^2 - z_2^2, \\ \langle R(X, W)W, X \rangle &= \frac{1}{4}\theta_2^2 - z_2^2, \quad \langle R(Y, Z)Z, Y \rangle = \frac{1}{4}\theta_1^2 - z_2^2, \\ \langle R(Y, W)W, Y \rangle &= \frac{1}{4}\theta_2^2 - z_2^2, \quad \langle R(Z, W)W, Z \rangle = 2z_2^2. \end{aligned}$$

One immediate consequence is that

$$Ric(X, X) = -\frac{1}{2}(\theta_1^2 + \theta_2^2) - 6z_2^2, \quad Ric(Z, Z) = \frac{1}{2}\theta_1^2.$$

showing that none of these Riemannian Lie groups is an Einstein manifold.

4. THE STRUCTURES J_1 AND J_2 ARE BOTH INTEGRABLE

It is easily seen that J_1 and J_2 are *both* integrable if and only if

$$2z_1 - z_4 - w_2 = 2z_2 + z_3 + w_1 = 0,$$

$$2z_1 + z_4 + w_2 = 2z_2 - z_3 - w_1 = 0.$$

As a direct consequence, we see that in this case the foliation \mathcal{F} is totally geodesic i.e.

$$z_1 = z_2 = z_3 + w_1 = z_4 + w_2 = 0.$$

In this situation the Lie bracket relations take the following form

$$\begin{aligned} [W, Z] &= \lambda W, \\ [Z, X] &= \alpha X + \beta Y - z_3 W, \\ [Z, Y] &= -\beta X + \alpha Y - z_4 W, \\ [W, X] &= aX + bY + z_3 Z, \\ [W, Y] &= -bX + aY + z_4 Z, \\ [Y, X] &= rX + \theta_1 Z + \theta_2 W. \end{aligned}$$

Example 4.1. For the non-vanishing coefficients α, β, θ_2 we have the following 3-dimensional family of Lie algebras

$$\begin{aligned} [W, Z] &= -2\alpha W, \\ [Z, X] &= \alpha X + \beta Y, \\ [Z, Y] &= -\beta X + \alpha Y, \\ [Y, X] &= \theta_2 W. \end{aligned}$$

They are the special cases $\mathfrak{g}_3(\alpha, \beta, 0, 0, \theta_2)$ of Example 4.3 in [5]. For the corresponding Riemannian Lie groups the Levi-Civita connection is given by

$$\begin{aligned} \nabla_X X &= \alpha Z, \quad \nabla_X Y = -\frac{1}{2}\theta_2 W, \quad \nabla_X Z = -\alpha X, \quad \nabla_X W = \frac{1}{2}\theta_2 Y, \\ \nabla_Y X &= \frac{1}{2}\theta_2 W, \quad \nabla_Y Y = \alpha Z, \quad \nabla_Y Z = -\alpha Y, \quad \nabla_Y W = -\frac{1}{2}\theta_2 X, \\ \nabla_Z X &= \beta Y, \quad \nabla_Z Y = -\beta X, \quad \nabla_Z Z = 0, \quad \nabla_Z W = 0, \\ \nabla_W X &= \frac{1}{2}\theta_2 Y, \quad \nabla_W Y = -\frac{1}{2}\theta_2 X, \quad \nabla_W Z = -2\alpha W, \quad \nabla_W W = 2\alpha Z. \end{aligned}$$

Using these identities it is easily seen that

- i) the Hermitian structure J_1 is Kähler if and only if $\theta_2 = -2\alpha$,
- ii) the Hermitian structure J_2 is Kähler if and only if $\theta_2 = 2\alpha$.

This means that in all the cases that we are considering at least one of the Hermitian structures J_1 or J_2 is not Kähler. For the sectional curvatures have

$$\begin{aligned}\langle R(X, Y)Y, X \rangle &= -\alpha^2 - \frac{3}{4}\theta_2^2, & \langle R(X, Z)Z, X \rangle &= -\alpha^2, \\ \langle R(X, W)W, X \rangle &= \frac{1}{4}\theta_2^2 - 2\alpha^2, & \langle R(Y, Z)Z, Y \rangle &= -\alpha^2, \\ \langle R(Y, W)W, Y \rangle &= \frac{1}{4}\theta_2^2 - 2\alpha^2, & \langle R(Z, W)W, Z \rangle &= -4\alpha^2.\end{aligned}$$

Finally, we yield

$$Ric(X, X) = -\frac{1}{2}\theta_2^2 - 4\alpha^2, \quad Ric(Z, Z) = -6\alpha^2, \quad Ric(W, W) = \frac{1}{2}\theta_2^2 - 8\alpha^2$$

telling us that if $4\alpha^2 \neq \theta_2^2$ then our Riemannian Lie group is not an Einstein manifold.

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